OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

COLLEGE OF ENGINEERING AND INFORMATICS

ENGINEERING MATHS QUALIFYING EXAMINATION 2021

First Paper

Time allowed: *Two* hours

Candidates for Computer Science & Information Technology and Project & Construction Management should take **4** questions out of 6. All other candidates should take **5** questions out of 6.

Formulae and Tables booklets are provided by the Exams Office
Calculators are permitted

1. (a) i. If 13 and 49 are the third and seventh terms respectively in an arithmetic sequence then find the sum to 20 terms of the same sequence.

   ii. The numbers $\frac{t}{4}$, t and $4t$ are the first, third and fifth terms respectively in a geometric series for which the sum of the fourth and fifth terms is 30 and the common ratio is *positive*; find the value of t.

   iii. For this value of t, show that $8t$ is the sixth term of each of the sequences in parts (i) and (ii).

   (b) A snail is trapped at the bottom of a well, and starts to climb vertically upwards. Initially it climbs 50 centimetres, then needs to rest. It then climbs 25 centimetres before resting again. It carries on like this, each climbing session covering half the vertical distance of the previous session. Show that the snail will never escape if the well is more than one metre deep.

2. (a) An object travels in such a way that its displacement in metres from a fixed origin is represented by the function

$$s(t) = t^3 - 7t + 6$$

   for $t \geq 0$, where t is measured in seconds.

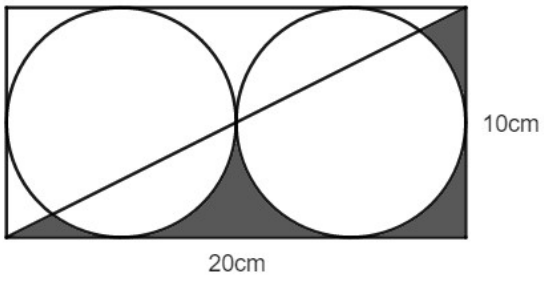
   i. What is the initial displacement?

   ii. At what time is the object's speed equal to -4m/s? What is the object's displacement at this instant?

   iii. Determine the time interval for which the displacement is negative.

   iv. At what time does the object reach its maximum negative displacement?

(b) A rectangle of width 20cm and height 10cm is just large enough to contain two touching circles as shown in the diagram below. Determine the combined area of the shaded regions.



3. (a) Differentiate the following functions with respect to x:

   i. $f(x) = e^{5x} + \cos x - \sqrt{x+1} + e^{\pi}$;

   ii. $g(x) = x^2 \log_e x$;

   iii. $h(x) = \dfrac{e^x}{\sin x}$.

   (b) Show that $y = \tan x$ is a solution to the equation $\dfrac{dy}{dx} = 1 + y^2$.

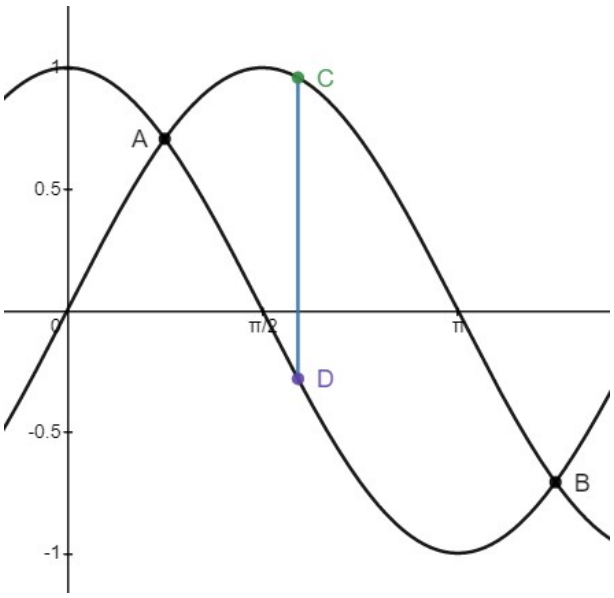
   (c) The normal to the parabola $y = x^2 + 2x - 1$ at the point $P(1, 2)$ crosses the x-axis at the point X. Find the area of the triangle OPX, where O is the origin $(0, 0)$.

4. (a) Find the following indefinite integrals.

   i. $\displaystyle\int 4e^{7x} - 2\sin(3x) + \frac{1}{3x+2}\, dx$

   ii. $\displaystyle\int (x + \frac{1}{x})(x - \frac{1}{x})\, dx$

   (b) The diagram below shows parts of the graphs of the sine and cosine functions. Points C and D are chosen to lie on these graphs between their points of intersection A and B in the region shown, in such a way that the line segment CD is vertical. Show that the length of the longest such line segment CD is numerically equal to half the area enclosed between the sine and cosine curves between A and B.



5. (a) Prove that $\cos^4 x - \sin^4 x = \cos 2x$.

   (b) Water is dripping from a hole in the base of a cylinder of radius r cm, where the water height is h cm, at a rate of 0.3 cm^3 s^{-1}.

      i. Fnd an expression for $\dfrac{dh}{dt}$, the rate at which the water level falls in the cylinder.

      ii. Hence find the rate of change in the water level, per minute, in a cylinder of radius 6 cm when the height of the water is 4 cm.

6. (a) Let $u = -2i$ and $v = 1 + i$, where $a, b \in \mathbb{R}$ and $i^2 = -1$, and let $z = u/v$.

      i. Write z in the form $r(\cos\theta + i\sin\theta)$, where $r, \theta \in \mathbb{R}$.

      ii. What is the angle between z and v?

      iii. Using de Moivre's theorem, or otherwise, find z^5.

   (b) Let $a, b, c, d \in \mathbb{R}$ and $z \in \mathbb{C}$. The sum of the roots of the general cubic equation $az^3 + bz^2 + cz + d = 0$ is $-b/a$, while the product of the roots is $-d/a$.

      i. Show that $z_1 = -1$ is a root of $z^3 + z^2 + z + 1 = 0$.

      ii. For the remaining roots z_2 and z_3, show that $z_2 = -z_3$.

      iii. Hence or otherwise show that $z_1^2 + z_2^2 + z_3^2 = -1$.